英

令和３年度 秋季募集
東北大学大学院機械系４専攻入学試験

試験問題冊子

数学Ａ　MATHEMATICS A

令和３年８月24日(火)
Tuesday, August 24, 2021　　9:30 − 10:30

Notice

1. Do not open this test booklet until instructed to do so.
2. A test booklet, answer sheets, and draft sheets are provided. Put your examinee number on each of the answer sheets, the draft sheets.
3. Answer all problems. Use two answer sheets for each problem.
4. At the end of the examination, double-check your examinee number and the problem numbers on the answer sheets. Put your answer sheets in numerical order on top of the other sheets, place them beside the test booklet, and wait for collection by an examiner. Do not leave your seat before instructed to do so by the examiner.

数 学 A　MATHEMATICS A

1. Solve the following problems, where a is a positive real number.

(1) Evaluate the following integral

$$\int_0^\infty re^{-r^2}\,dr.$$

(2) Evaluate the following integral

$$\int_0^\infty \int_0^\infty e^{-a\left(x^2+y^2\right)}\,dxdy.$$

(3) Let us define $F_n(a)$ by

$$F_n(a) = \int_0^\infty x^{2n}e^{-ax^2}\,dx,$$

where $n \geq 0$ is an integer.

a) Evaluate $F_0(a)$.

b) Evaluate $F_1(a)$.

数学 A　MATHEMATICS A

2. The matrix A is given by

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 2 & 1 \\ 3 & 5 & 2 \end{pmatrix}.$$

Solve the following problems, where a is a real number.

(1) Find the eigenvalues of A.

(2) Obtain $A^3 - 5A^2 + aA$.

(3) Find a such that $A^3 - 5A^2 + aA$ has a non-zero eigenvalue with multiplicity two.

3. In the Cartesian coordinate system (x, y, z), the vector field $\boldsymbol{A}$ is given by

$$\boldsymbol{A} = (yz - y)\,\boldsymbol{i} + (xz - 3x)\,\boldsymbol{j} + (z^2 + xy)\,\boldsymbol{k},$$

where $\boldsymbol{i}$, $\boldsymbol{j}$, and $\boldsymbol{k}$ are the fundamental vectors in the x, y, and z directions, respectively. In addition, the hyperboloid S is given by

$$S:\ z = \sqrt{x^2 + y^2 - 1}, \qquad 0 \leq z \leq 2,$$

and C is the boundary of S at $z = 2$. Solve the following problems.

(1) Obtain $\nabla \times \boldsymbol{A}$.

(2) Evaluate the line integral

$$\int_C \boldsymbol{A} \cdot d\boldsymbol{r},$$

where $\boldsymbol{r}$ is the position vector on C and the line integral is taken in the rotation direction of the right-handed screw advancing in the $+z$ direction.

(3) Evaluate the surface integral

$$\int_S (\nabla \times \boldsymbol{A}) \cdot \boldsymbol{n}\, dS,$$

where $\boldsymbol{n}$ is the unit normal vector of S and the z component of $\boldsymbol{n}$ is negative or zero. (Hint: use Stokes' theorem if necessary.)

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試験問題冊子

数学Ｂ MATHEMATICS B

令和３年８月24日(火)
Tuesday, August 24, 2021 13:00 − 14:00

Notice

1. Do not open this test booklet until instructed to do so.
2. A test booklet, answer sheets, and draft sheets are provided. Put your examinee number on each of the answer sheets, the draft sheets.
3. Answer all problems. Use two answer sheets for each problem.
4. At the end of the examination, double-check your examinee number and the problem numbers on the answer sheets. Put your answer sheets in numerical order on top of the other sheets, place them beside the test booklet, and wait for collection by an examiner. Do not leave your seat before instructed to do so by the examiner.

数学 B　MATHEMATICS B

1. Find the general solutions of the following ordinary differential equations.

(1) $$\begin{cases} \dfrac{dy}{dx} + 2z = \cos x \\ \dfrac{dz}{dx} - 2y = -\sin x \end{cases}$$

(2) $y\sqrt{1+y^2} - 2x\dfrac{dy}{dx} = 0$

2. The Fourier series of a function $f(x)$ $(-\pi \leq x < \pi)$ with period 2π is defined by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \{ a_n \cos(nx) + b_n \sin(nx) \} .$$

Solve the following problems.

(1) Show

$$\frac{1}{\pi}\int_{-\pi}^{\pi} \{ f(x) \}^2 \, dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2) .$$

(2) Obtain the Fourier series of $f(x) = x$ $(-\pi \leq x < \pi)$ with period 2π, and show

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} .$$

数学B　MATHEMATICS B

3. The Laplace transform of a function $f(t)$ is defined by

$$\mathcal{L}[f(t)] = F(s) = \int_0^\infty f(t)\, e^{-st}\, dt.$$

Solve the following problems.

(1) Let $f(t) = r^n$ in the interval of $n \le t < n+1$ $(n = 0, 1, 2, \cdots)$, as shown in Fig. 1 on the next page. Obtain the Laplace transform of $f(t)$, where r is a real number and you may use

$$\sum_{n=0}^{\infty} r^n e^{-ns} = \frac{1}{1 - re^{-s}}.$$

(2) For a real sequence $\{a_n\}$, the function $g(t)$ is given by $g(t) = a_n$ in the interval of $n \le t < n+1$ $(n = 0, 1, 2, \cdots)$. Express $\mathcal{L}[g(t+1)]$ with $G(s)$ and a_0, where $G(s)$ is the Laplace transform of $g(t)$.

(3) The sequence $\{a_n\}$ $(n = 0, 1, 2, \cdots)$ satisfies the following relation

$$a_{n+1} - 3a_n - 2^{n+1} = 0.$$

When $a_0 = 4$, obtain the general term of a_n using the results of problems (1) and (2).

$$\left(\text{Hint}: \frac{e^{-s}}{(1-2e^{-s})(1-3e^{-s})} = \frac{-1}{1-2e^{-s}} + \frac{1}{1-3e^{-s}}.\right)$$

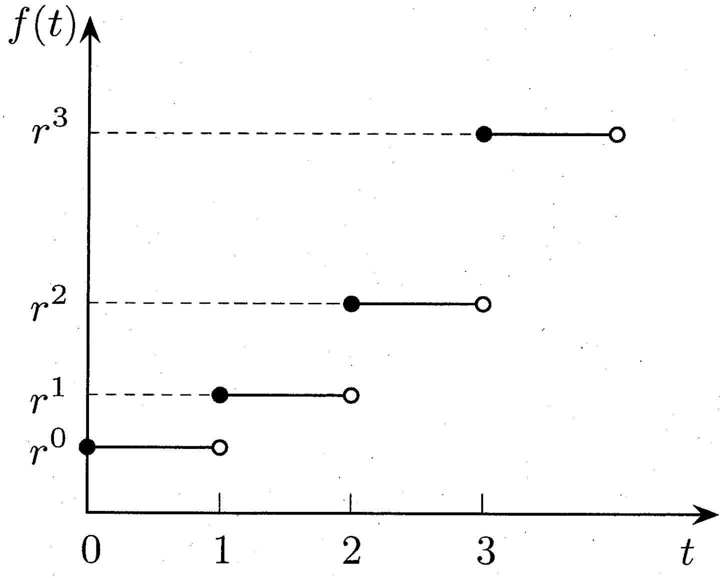

Fig. 1

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試験問題冊子
【専門科目　Specialized Subjects】

熱力学	THERMODYNAMICS	P1~2
流体力学	FLUID DYNAMICS	P3~4
材料力学	STRENGTH OF MATERIALS	P5~6
機械力学	DYNAMICS OF MECHANICAL SYSTEMS	P7~8
制御工学	CONTROL ENGINEERING	P9~10

令和３年８月24日（火）　15:45 － 16:45
Tuesday, August 24, 2021　15:45 － 16:45

Notice

1．Do not open this test booklet until instructed to do so.

2．A test booklet, answer sheets, draft sheets, and two selected-subjects forms are provided. Put your examinee number on each of the answer sheets, the draft sheets, and the form.

3．Select two subjects from the five subjects in the booklet and answer all problems in each subject. Indicate your selection on the selected-subjects form. Use one set of two answer sheets for each subject, and use one sheet per problem.

4．At the end of the examination, double-check your examinee number and the problem numbers on the answer sheets. Put your answer sheets in numerical order on top of the other sheets, place them beside the test booklet, and wait for collection by an examiner. Do not leave your seat before instructed to do so by the examiner.

熱 力 学 THERMODYNAMICS

1. Consider a system of an adiabatic container of volume V filled with an ideal gas. The ideal gas has specific heat at constant pressure c_p and is divided into regions 1 and 2 by an adiabatic partition. The mass of the ideal gas in region 1 is equal to that in region 2, which is denoted by m. The pressure in region 1 is equal to that in region 2, which is denoted by p. The temperatures in the regions 1 and 2 are T_1 and T_2, respectively. When the partition is removed, the ideal gases are mixed together. The volume of the partition can be neglected. Answer the following questions.

(1) Obtain the temperature of the system after the mixing.

(2) Using the equation of state, show that the pressure of the system does not change before and after the mixing.

(3) Obtain the entropy change of the system before and after the mixing.

熱 力 学　THERMODYNAMICS

2. Answer the following questions, where p, s, T, u and v are pressure, specific entropy, temperature, specific internal energy and specific volume, respectively.

(1) Derive the following Maxwell thermodynamic relation using the equation of specific Helmholtz free energy, $f = u - Ts$.

$$\left(\frac{\partial s}{\partial v}\right)_T = \left(\frac{\partial p}{\partial T}\right)_v$$

(2) Derive the following equation.

$$\left(\frac{\partial u}{\partial v}\right)_T = T\left(\frac{\partial p}{\partial T}\right)_v - p$$

(3) Show that the specific heat at constant volume of an ideal gas is a constant or a function of temperature only, by using the equation in question (2).

流 体 力 学　FLUID DYNAMICS

1. Consider a two-dimensional steady potential flow of an inviscid incompressible fluid. The complex velocity potential $W(z)$ of the flow is given by

$$W(z) = C(1-i)\log z,$$

where C is a positive real number and log is the natural logarithm. The complex variable z is given by $z = re^{i\theta}$, where r and θ are the radial and circumferential coordinates, respectively, and $i = \sqrt{-1}$. Answer the following questions.

(1) Obtain the velocity potential $\phi(r,\theta)$ and the stream function $\psi(r,\theta)$ of the flow field.

(2) Obtain the radial velocity V_r and the circumferential velocity V_θ.

(3) Obtain the pressure p_1 at $r = 1$. Here, the density of the fluid is constant at ρ and the pressure at infinity is p_∞.

(4) Choose an appropriate figure illustrating the streamlines of the flow field from among Figs. 1 a) - d).

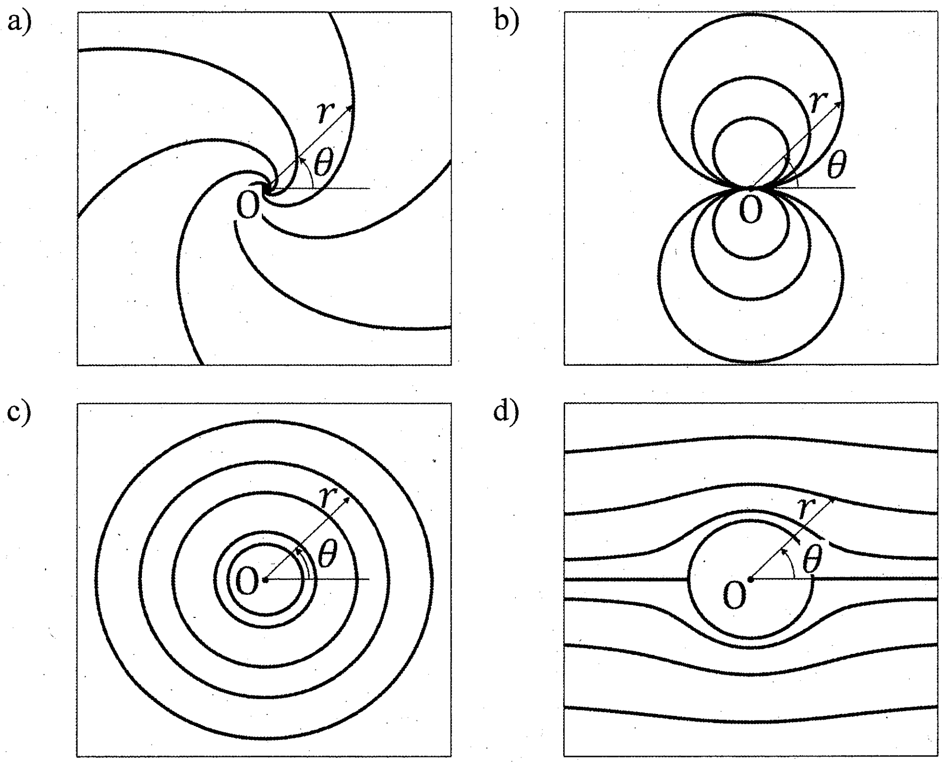

Fig. 1

2. Water filled in a large container flows out through a circular pipe system as shown in Fig. 2. A circular pipe with the cross-sectional area S_a is horizontally connected to the container at the vertical distance H from the water surface. A coaxial circular pipe with the enlarged cross-sectional area S_b ($S_b > S_a$) is connected to the horizontal circular pipe. There is a water column at the enlarged part. The height of the water column from the central axis of the circular pipe is d. The cross-sectional area of the downstream pipe connected to the enlarged pipe is S_a. The pipe bends vertically, then the water issues with the velocity V at the vertical distance h from the central axis of the horizontal pipe. The flow in the pipe system is steady, and the pressure surrounding the container and the pipe is p_0, the pressure at the depth H in the container is p_1, the pressure in the horizontal pipe connected with the container is p_2, the pressure in the enlarged pipe is p_3. Assuming that the container is sufficiently large, the water in the container is static and H is constant. The diameters of the pipe system are sufficiently small compared with the distances H, d and h, and the pressure and the velocity at each cross-sectional plane in the circular pipe are uniform. The density of water ρ is constant. The gravitational acceleration is g. Any loss in the pipe system is negligible. Answer the following questions.

(1) Express p_1 using p_0, ρ, g and H.

(2) Express the velocity v in the enlarged part of the pipe using S_a, S_b and V.

(3) Express V using g, H and h.

(4) Show the relation of magnitudes among p_0, p_1, p_2 and p_3 in descending order using a greater-than symbol.

(5) Express the ratio of the cross-sectional area of the circular pipe to that of the enlarged pipe S_a / S_b using H, d and h.

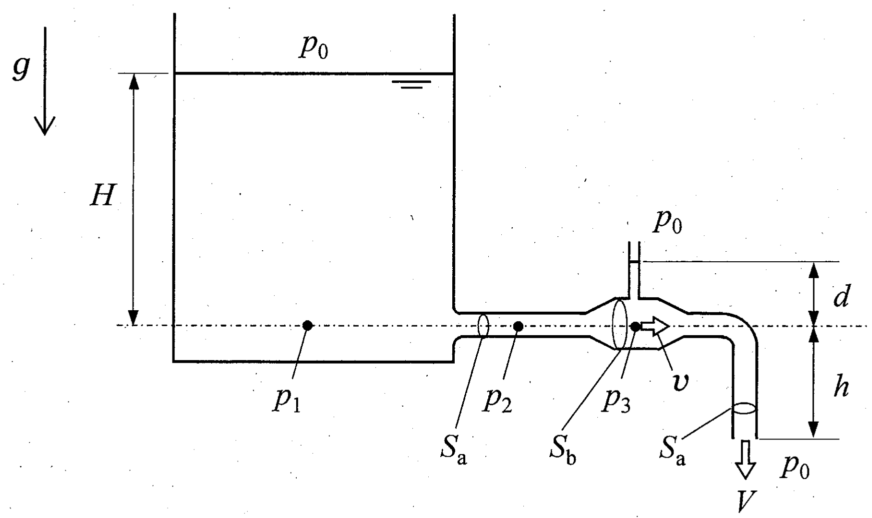

Fig. 2

材 料 力 学　STRENGTH OF MATERIALS

1. Consider a truss composed of three uniform bars AD, CD and BD, as shown in Fig. 1. The bar CD with length L is vertically pinned to a horizontal rigid ceiling. The bars AD and BD are length $L / \cos\theta$ and pinned to the rigid ceiling at one end and pinned to bar CD at point D, the other end. The angle of the bar axis AD and BD from the CD is θ. All the bars are assumed to be stress free. The Young's modulus, cross-sectional area and coefficient of thermal expansion of three bars are E, A and $\alpha\,(>0)$, respectively. Neglect the weight of the bars. When only the temperature of the bar CD is elevated by ΔT, answer the following questions.

   (1) Denote the axial stresses in bars AD, CD and BD by σ_{AD}, σ_{CD} and σ_{BD}, respectively. Derive the force balance equations in vertical and horizontal directions.
   (2) Find the relationship among σ_{AD}, σ_{CD} and σ_{BD}.
   (3) Show the axial displacements of bar δ_{AD}, δ_{CD} and δ_{BD} using σ_{AD}, σ_{CD} and σ_{BD}.
   (4) Determine σ_{AD}, σ_{CD} and σ_{BD}.

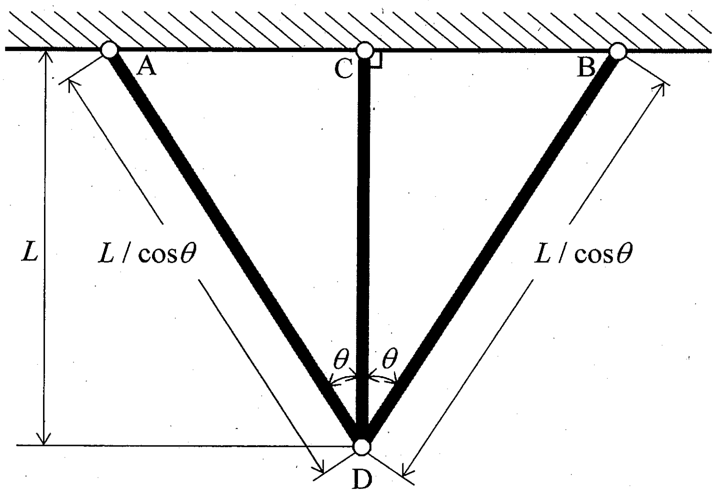

Fig. 1

2. A beam AB of length $2L$ is fixed to a vertical rigid wall at end A, as shown in Fig. 2(a). The flexural rigidity of the beam is EI. Neglect the weight of the beam. Direction of the bending moment is positive when the beam deforms convex downwards. A coordinate x is taken along the neutral axis of the beam and its origin is located at point A, as shown in Fig. 2(a). The beam is subjected to an external load W and a bending moment M_0. The bending moment diagram (BMD) of the beam is shown in Fig. 2(b). Answer the following questions.

(1) Draw the direction and position of W and M_0 applied to beam AB with a figure of the beam.
(2) Express the bending moment applied to beam AB as a function of x.
(3) Determine the deflection of the beam at its right end B.

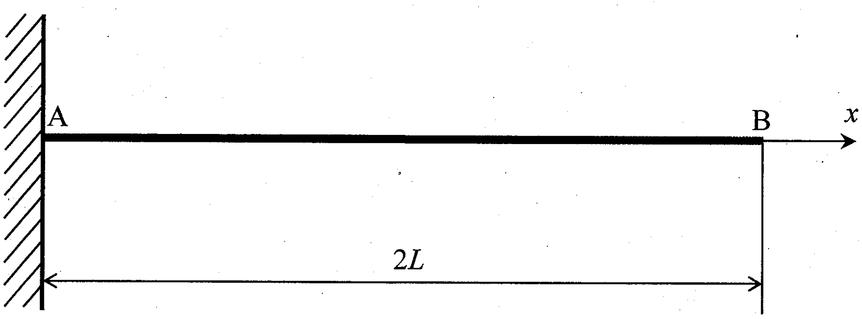

Fig. 2(a)

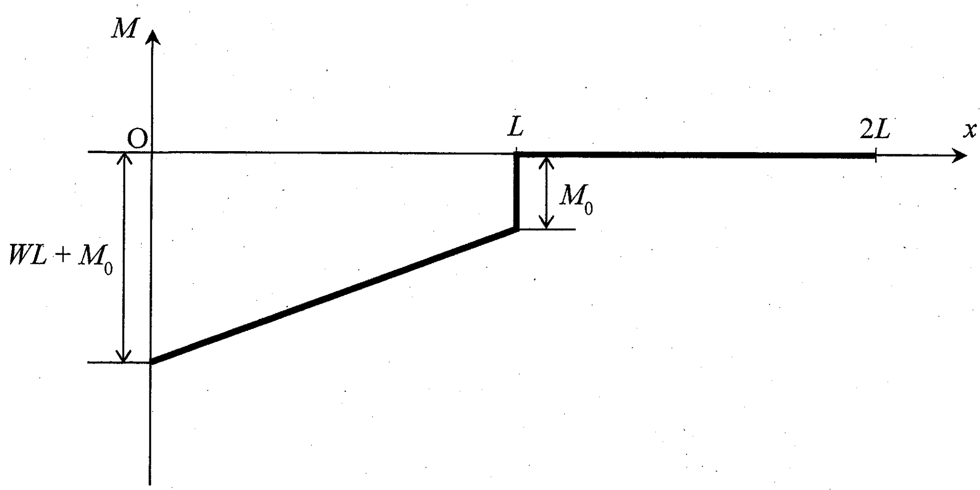

Fig. 2(b)

機械力学 DYNAMICS OF MECHANICAL SYSTEMS

1. Consider a vibration system consisting of a mass m, a spring with spring constant k, and a dashpot with damping coefficient c as shown in Fig. 1. Oscillation force $F(t) = P\sin\omega t$ is applied to the mass, where P and ω are the amplitude and the angular frequency of the force, respectively, and t is time. When $c < \sqrt{2mk}$ and the system is in a steady state, answer the following questions.

(1) Derive the equation of motion of the system.

(2) The displacement x of the mass m is given by

$$x = A\sin(\omega t - \phi),$$

where

$$A = \frac{P}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}} \qquad \text{and} \qquad \phi = \tan^{-1}\frac{c\omega}{k - m\omega^2}.$$

Obtain ω which maximizes the amplitude A.

(3) Obtain $A_{\max}/A_{\mathrm{s}}$, where $A_{\max}$ is the maximum value of A at ω obtained in question (2), and A_{s} is the static displacement of the mass m when a constant force P is applied.

(4) Show that $A_{\max}/A_{\mathrm{s}}$ approaches $\sqrt{mk}/c$ when $c \ll \sqrt{mk}$.

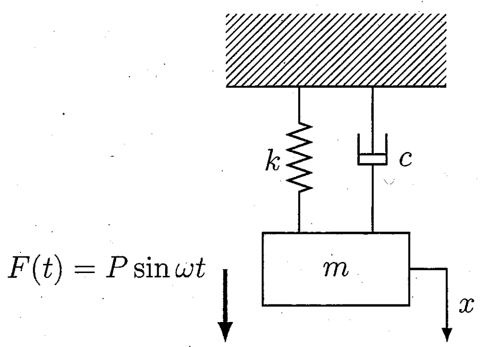

Fig. 1

2. Consider a system consisting of a cart with mass m, two springs with spring constants k_1, k_2, and a pendulum with a mass M and a string of length l, as shown in Fig. 2. The cart is connected to the walls via the two springs and can move only in the horizontal direction without friction. The pendulum is connected to the center of the mass of the cart and can rotate only in the plane of the figure. x is the horizontal displacement of the cart from the equilibrium position. θ is the angular displacement of the pendulum from the vertical direction. The gravitational acceleration is g. The masses of the string and the springs are negligible. Answer the following questions.

(1) Derive the kinetic energy T of the system.

(2) Derive the potential energy U of the system.

(3) Derive the equations of motion of the system by using the Lagrange equation.

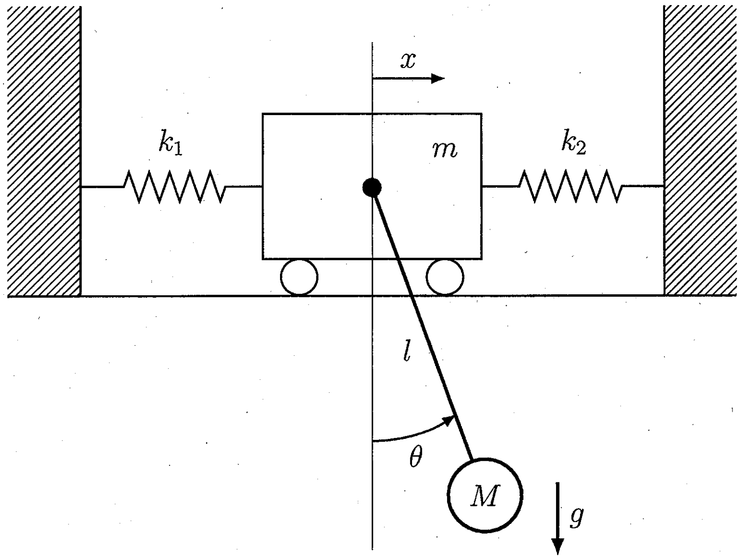

Fig. 2

制御工学 CONTROL ENGINEERING

1. Solve the following problems.

   (1) Consider the linear time-invariant system $P(s)$, which has gain characteristics shown in Fig. 1. The gain characteristics are approximated using line segments and have gain approximately 0 dB at $\omega = 1$ rad/s.

   a) Find the transfer function $P(s)$.

   b) Find the impulse response of $P(s)$.

   c) Find the stationary output $y(t)$ when the input is $u(t) = \sin t$.

   (2) Consider the feedback control system shown in Fig. 2. The transfer functions of the plant $P(s)$ and the controller $C(s)$ are given by

$$P(s) = \frac{1}{s(s+2)}, \quad C(s) = K\frac{Ts+1}{0.1s+1}, \quad K > 0, \quad \text{and} \quad T > 0.$$

   Find the range of T over which the system is stable no matter how large the gain K is.

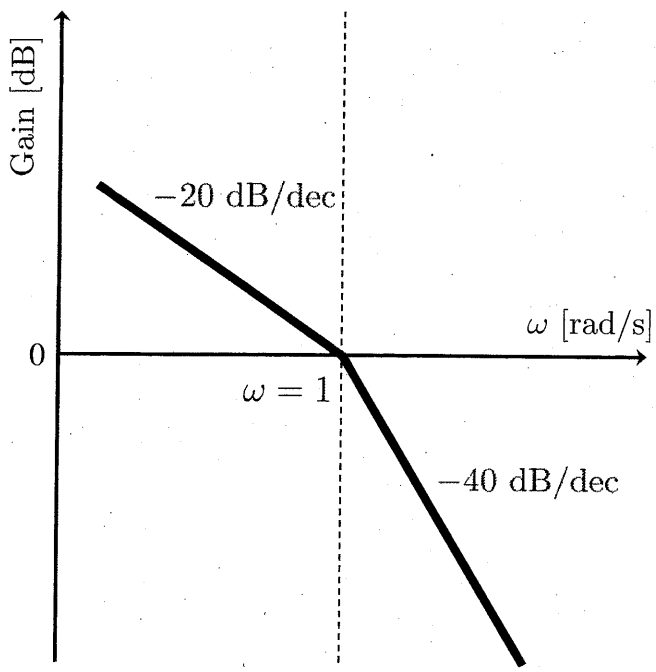

Fig. 1

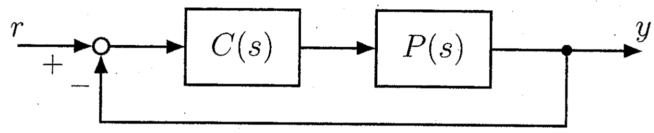

Fig. 2

2. Consider the system represented by

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{b}u(t), \quad y(t) = \boldsymbol{c}\boldsymbol{x}(t),$$

where $u(t)$ is the input, $y(t)$ is the output, $\boldsymbol{x}(t) = [x_1(t) \quad x_2(t)]^T$ is the state vector, and

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \text{and} \quad \boldsymbol{c} = [1 \quad 2].$$

Solve the following problems.

(1) Check if this system is controllable or not.

(2) When the output feedback $u(t) = -ky(t)$ is applied, find the range of k for the closed-loop system to be stable.

(3) Consider the system given in problem (2) with $k = 2$. Discuss the stablility of the closed-loop system using the following Lyapnov function candidate

$$V(t) = \boldsymbol{x}(t)^T \boldsymbol{P}\boldsymbol{x}(t), \quad \text{and} \quad \boldsymbol{P} = \begin{bmatrix} 31 & 9 \\ 9 & 4 \end{bmatrix}.$$